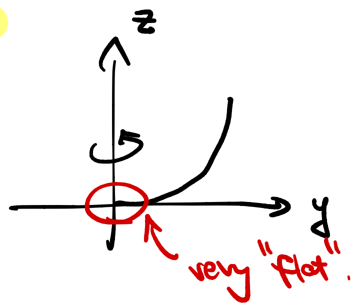
Geometric meaning of Gaussian curvature.

From Midterm:

① $S_p \equiv 0, \forall p \nearrow$ planar
$\searrow K(p) \equiv 0$

But $S_p = 0$ at some $p \in S \not\Rightarrow$ planar.

Example:



$z = y^4$.

S parametrised by
$(u, v, (u^2+v^2)^2)$

at $p = (0,0,0)$, $N_p = (1,0,0)$ (fixing orientation)

$$\begin{cases} X_u = (1, 0, 4u(u^2+v^2)) \\ X_v = (0, 1, 4v(u^2+v^2)) \end{cases} \Rightarrow X_{uu} = X_{uv} = X_{vv} = 0 \text{ at } p$$

$$\Downarrow$$

$$\mathrm{II}_p \equiv 0 \ \#.$$

But $S \neq$ plane!!

What is curvature ??

Prop: Let $p \in S$ with $K(p) \neq 0$. Let B_n be a seq. of open sets s.t. $\sup_{q \in B_n} |q - p| \to 0$ as $n \to \infty$.

Then $$\lim_{n\to\infty} \frac{\text{Area}(N(B_n))}{\text{Area}(B_n)} = |K(p)|$$

where $N : S \to S^2$ is the gauss map.

pf: Let $X: \mathcal{U} \to S$ be a local parametrization of S at p. Let $X(\mathcal{U}_n) = B_n$, then

- $\text{Area}(B_n) = \int_{U_n} \sqrt{\det(g)}\, du\, dv$
- $\text{area}(N(B_n)) = \int_{N(B_n)} dA$ where $N(B_n) \subseteq S^2$.

now $N \circ X$ is a local parametrization of $N(S)$ s.t.

$$\text{area} = \int_{B_n} \| N_u \times N_v \|\, du\, dv. \quad \text{where}$$

$$N_u = S_p(X_u) = S_p(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \lambda_1 u_1 + \alpha_2 \lambda_2 u_2$$

$$\Rightarrow N_u \times N_v = (\lambda_1 \alpha_1 u_1 + \lambda_2 \alpha_2 u_2) \times (\lambda_1 \beta_1 u_1 + \lambda_2 \beta_2 u_2)$$

$$= (\lambda_1 \lambda_2 \alpha_1 \beta_2 - \lambda_1 \lambda_2 \alpha_2 \beta_1)\, u_1 \times u_2$$

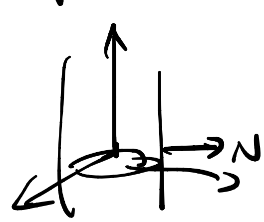
$$\therefore \| N_u \times N_v \| = |K(p)| \left\| \det \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \right\| \quad (\{u_1, u_2\} = \text{o.n.})$$

$$= |K(p)| \cdot \| X_u \times X_v \|.$$

$$\therefore \text{Ratio} = \frac{\int_{B_n} |K| \cdot \| X_u \times X_v \|}{\int_{B_n} \| X_u \times X_v \|} \to |K(p)| \text{ as } n \to \infty. \quad \#.$$

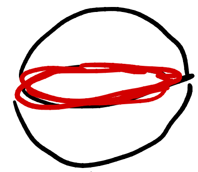
Eg 1: $S = \text{plane} \Rightarrow S_p = 0 \Rightarrow K(p) = 0$

2: $S = S \times \mathbb{R}$



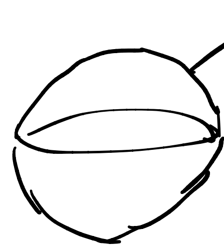
$$S_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad N(S) = S^1 \subseteq S^2.$



and $\quad \text{Area}(N(S)) = 0$ (lower dimension)

3: $\quad S = S^2$



$\Rightarrow \quad N(p) = p \quad (\text{or } -p)$

$$\Rightarrow \begin{cases} K(p) = 1. \\ \text{Area}(N(S)) = \text{Area}(S) = 4\pi. \end{cases}$$

4: Torus

$$X(u,v) = \left((a+r\cos u)\cos v,\ (a+r\cos u)\sin v,\ r\sin u \right)$$

$$\Rightarrow \quad X_u = (-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u)$$

$$X_{uu} = (-r\cos u\cos v,\ -r\cos u\sin v,\ -r\sin u)$$

$$X_{uv} = (r\sin u\sin v,\ -r\sin u\cos v,\ 0)$$

$$X_v = (-(a+r\cos u)\sin v,\ (a+r\cos u)\cos v,\ 0)$$

$$X_{vv} = (-(a+r\cos u)\cos v,\ -(a+r\cos u)\sin v,\ 0)$$

$$\Rightarrow \quad [g]_X = \begin{bmatrix} r^2 & 0 \\ 0 & (a+r\cos u)^2 \end{bmatrix} \quad \text{and} \quad \sqrt{\det[g]_X} = r(a+r\cos u).$$

Recall :

$$e = \frac{\langle X_u \times X_v, X_{uu} \rangle}{\sqrt{EG-F^2}}$$

$$= \frac{1}{\cancel{r}\cancel{(a+r\cos u)}} \cdot \begin{vmatrix} -\cancel{r}\sin u\cos v & -\cancel{r}\sin u\sin v & \cancel{r}\cos u \\ -\cancel{(a+r\cos u)}\sin v & \cancel{(a+r\cos u)}\cos v & 0 \\ -r\cos u\cos v & -r\cos u\sin v & -r\sin u \end{vmatrix}$$

$$= -r \begin{vmatrix} -\sin u\cos v & -\sin u\sin v & \cos u \\ -\sin v & \cos v & 0 \\ \cos u\cos v & \cos u\sin v & \sin u \end{vmatrix}$$

$$= -r \left(\cos u \cdot \begin{vmatrix} -\sin v & \cos v \\ \cos u\cos v & \cos u\sin v \end{vmatrix} + \sin u \begin{vmatrix} -\sin u\cos v & -\sin u\sin v \\ -\sin v & \cos v \end{vmatrix} \right) = r.$$

Similar to e, f, g

$$\Rightarrow K(p) = \frac{\cos u}{r(a+r\cos u)} \qquad \text{while}$$

$$\int_S K\,dA = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos u}{r(a+r\cos u)} \cdot r(a+r\cos u)\,du\,dv$$

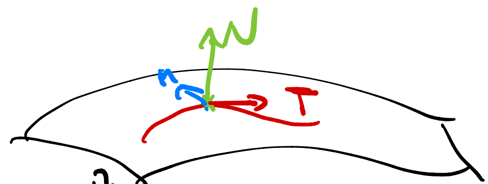
$$= 0. \quad \#.$$

Recall :

★ The 2nd f.f. $\mathrm{II}(v,w) = -\langle dN_p(v), w\rangle$, $\forall v, w \in T_pS$, $p \in S$.

Consider a C^∞ curve $\alpha: (-\varepsilon,\varepsilon) \to S \subseteq \mathbb{R}^3$.

Then $\alpha' = T$



taking o.n. frame $\{T, n, N\}$ s.t. $n = N \times T$.

$\Rightarrow$ $T' = \alpha'' \in \operatorname{span}\{n, N\}$

i.e. $\alpha''(s) = \underbrace{k_g(s)}_{\text{geodesic curvature}} \cdot n + \underbrace{k_n(s)}_{\text{normal curvature}} \cdot N$

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observe : If $|\alpha'| = 1$ (arc-length parametrized)

① If $k(s)$ = curvature of α $\left(\text{i.e. } k(s) = |\alpha''|\right)$,

then $k_g = \langle \alpha'', N\rangle = k\langle N_\alpha, N\rangle = k\cos\theta$

where θ = angle between N, α''.

② $k_n = \langle \alpha'', N\rangle = -\langle \alpha', N'\rangle$

$= \langle S_p(\alpha'), \alpha'\rangle = \mathrm{II}(\alpha', \alpha')$

$\Rightarrow$ prop :

Normal curve of a curve α on S, at $p = \alpha(0)$

depends only on $\alpha'(0)$.

Moreover $k_2 \le k_n \le k_1$ where k_i = principle curv.

More generally, let $v \in T_pS$ with $|v| = 1$.

$\Rightarrow$ $v = \cos\theta\, u_1 + \sin\theta\, u_2$ for some $\theta \in [0, 2\pi]$.

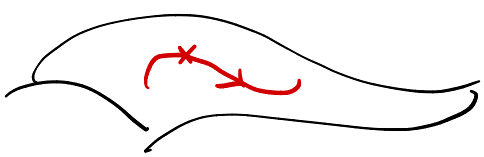
with eigen-vector u_1, u_2 of S_p.

$\Rightarrow$ $k_n(p) = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$ along α s.t. $\alpha'(0) = v$. at $p = \alpha(0)$

<u>Special curve</u>: If α is a regular curve on S

s.t. $\alpha'(s) \parallel$ principle direction, for all s

then α is called, a line of curvature of S.



$S_p(\alpha') = \lambda_t \alpha'$, $\forall t \in I$.

<u>prop</u>: Regular curve = line of curvature of S

iff $N'(t) = \lambda(t) \cdot \alpha'(t)$ for some differentiable fcn

Moreover, in this case $-\lambda(t)$ = principle curv. along $\alpha(t)$.

<u>pf</u>: $(\Rightarrow)$: $dN_p(\alpha') = \lambda(t) \cdot \alpha'(t)$ for some fcn λ.

$\Rightarrow$ λ must be differentiable since N, α' are C^∞ on t.

$(\Leftarrow)$ $dN_p(\alpha') = \lambda(t)\, \alpha'(t)$

$\Rightarrow$ $S_p(\alpha') = -\lambda(t) \cdot \alpha'(t)$. #.

Now, Study S using shape operator S_p

Special case: plane $\Rightarrow S_p \equiv 0$

sphere $\Rightarrow S_p = \lambda \, Id$.

In general, if $k_1 = k_2$,

then all direction = principle direction (direction of eigen-vect)

$\Rightarrow S_p = k \cdot Id$.

$\Rightarrow K_p = k^2$, $H_p = k$.

Plane: $k = 0$

sphere: $k = \pm 1$ (dep. orientation, size of radius)

prop: Suppose $X: \mathcal{U} \to S$ be a local parametrization of S. which is connected. If $S_p = \lambda_p \cdot Id$, $\forall p \in X(\mathcal{U})$ (umbilical) then $X(\mathcal{U}) \subseteq$ plane or sphere.

pf: $\lambda_p = \dfrac{\langle S_p(X_u), X_u \rangle}{\| X_u \|^2}$ is smooth in $p \in S$.

$\Rightarrow \lambda$ is a differentiable fcn on S

Midterm last Que: $\lambda(p) \equiv \lambda_0$, $\forall p \in X(\mathcal{U})$

by showing $\lambda_u = \lambda_v = 0$

Case 1: $\lambda_0 \equiv 0 \Rightarrow N \equiv N_0 \in S^2 \subseteq \mathbb{R}^3$.

$\Rightarrow \langle X - X_0, N_0 \rangle \equiv 0 \Rightarrow$ planar.

Case 2: $\lambda_0 \neq 0 \Rightarrow N_i = dN_p(X_i) = -\lambda_0 X_i$

$\Rightarrow (N + \lambda_0 X)_i = 0 \quad , \forall i = 1, 2$

$\Rightarrow N + \lambda_0 X = \text{const}$, called $\lambda_0 Y_0 \in \mathbb{R}^3$

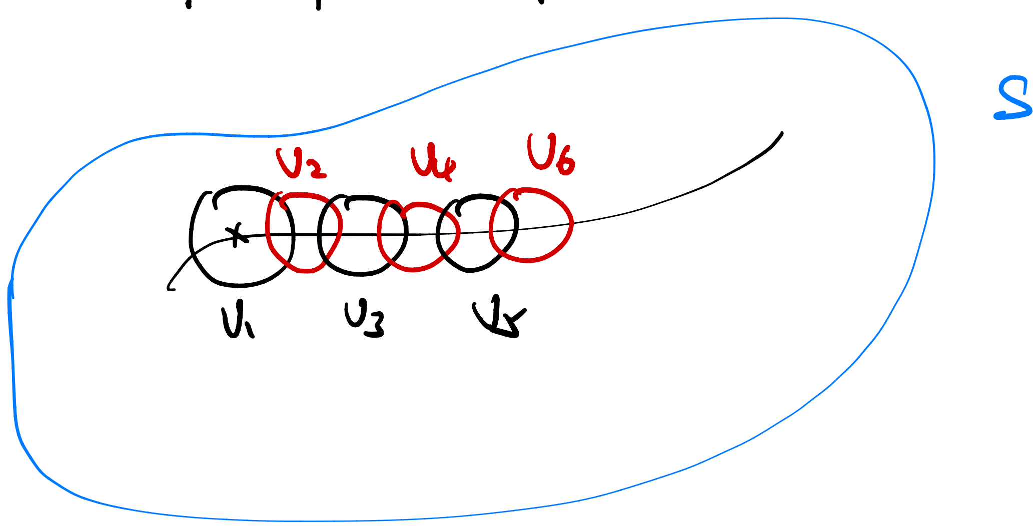
$\Leftrightarrow X - Y_0 = \lambda_0^{-1} N$

$\therefore X \in \underbrace{Y_0}_{\text{translation}} + \underbrace{\lambda_0^{-1}}_{\text{scaling}} S^2$ #.

In general,

Corollary: If S = regular surface, connected, s.t. $S_p = \lambda_p \cdot Id$, $\forall p \in S$, then same is true.

pf:



Case 1: $\lambda_p \equiv 0$ on U_1

let $\Omega = \{p \mid \lambda_p = 0\}$

- closed by defn.
- open by previous (topology)

$\Rightarrow \Omega = S$ #

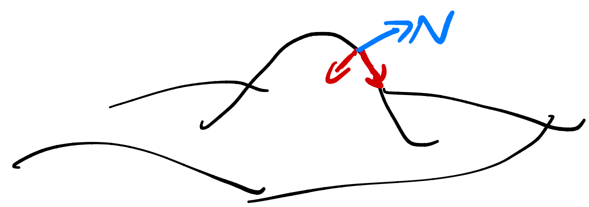
case 2: $\lambda_p \equiv \lambda_0 \neq 0$ is similar.

Local structure :

Let $p \in S$, $T_pS = \text{span}\{u_1, u_2\}$ where

$S_p(u_i) = \lambda_i u_i$ (the principle direction)

and $N = u_1 \times u_2$.



By rotation + translation

assume $\begin{cases} u_1 = (1,0,0) \\ u_2 = (0,1,0) \end{cases}$, $p = (0,0,0)$, $S \overset{\text{locally}}{=} \{(x,y,z) \mid z = f(x,y)\}$

Then under this coordinate.

prop:

$$f(x,y) = \frac{1}{2}(k_1 x^2 + k_2 y^2) + o(r^2)$$

$\Rightarrow$ locally ① elliptic paraboloid if S_p = elliptic

② hyperbolic paraboloid if S_p = hyperbolic

③ parabolic cylinder if S_p = parabolic

pf $\begin{cases} u_1 = (1,0,0) \\ u_2 = (0,1,0) \end{cases} \Rightarrow f_x(0,0) = f_y(0,0) = 0$

$$X_{ij} = (0, 0, f_{ij}) \quad \text{and} \quad N = \frac{X_1 \times X_2}{\| X_1 \times X_2 \|}$$

$$X_1 = (1, 0, f_x) \quad ; \quad X_2 = (0, 1, f_y)$$

$$\Rightarrow \quad N = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$\therefore \quad -dN_p(X_x) = S_p(X_x) = -N_x = (\lambda_1, 0, 0)$$

$$= \frac{(f_{xx}, f_{xy}, 0)}{\sqrt{1 + f_x^2 + f_y^2}}$$

$$+ -\frac{1}{2}\left(1 + f_x^2 + f_y^2\right)^{-3/2} \cdot (2 f_x f_{xx} + 2 f_y f_{xy})$$

$$(f_x, f_y, 0)$$

$$= (f_{xx}, f_{xy}, 0) \quad \text{at } p.$$

$$\therefore \begin{cases} f_{xy} = 0 & \text{at } p \\ f_{xx} = \lambda_1 \end{cases}$$

Similarly $\quad f_{yy} = \lambda_2$

$\therefore$ Taylor series $\Rightarrow \quad f = \frac{1}{2}(\underset{k_1}{\underset{\shortparallel}{\lambda_1}} x^2 + \underset{k_2}{\underset{\shortparallel}{\lambda_2^2}} y^2) + o(r^2). \quad \#$